Using GeoGebra for Correction of Incorrect Premises in Geometric Proofs

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Introduction

Mathematical proofs are often neglected in some cases, especially in primary schools. Let us admit, however, that their role is irreplaceable - among other things, they help to develop logical thinking and contribute correctly to a better understanding of the issue. At the same time, the lower age of pupils does not have to be a problem. One example is the well-known demonstration of the validity of the Pythagorean theorem, which almost everyone has encountered during their studies.

Recall that the proof (admitting at this point that it is not proof of a generally valid theorem "The square whose side is the hypotenuse in rectangular triangle is equal to the sum of the areas of the squares on the other two sides," rather than a visual demonstration which is quite accessible even for young pupils) of this theorem is based on a picture containing a pair of identical squares with their side lengths $a + b$.
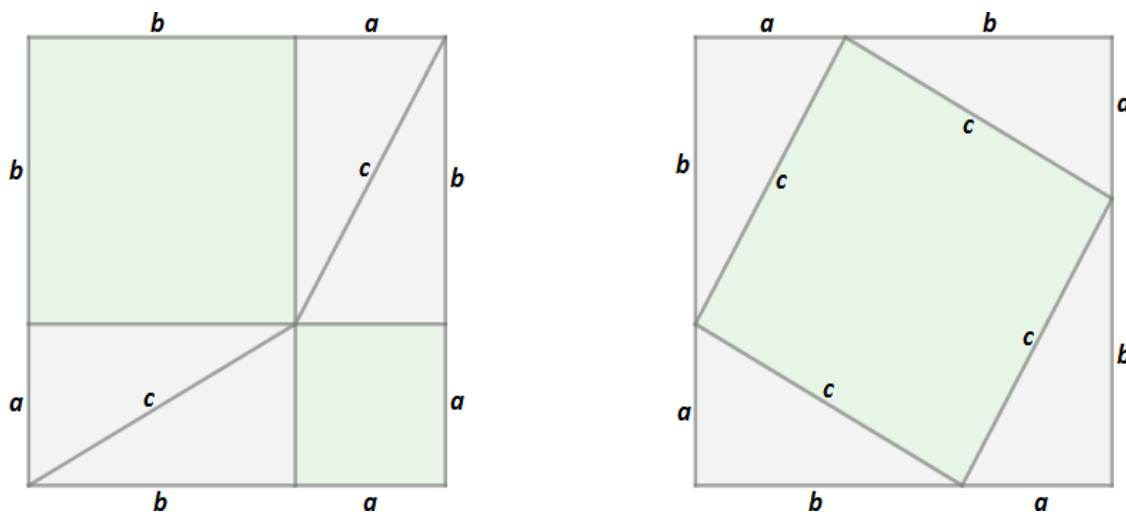

Figure 1. Graphical Proof of Pythagorean Theorem

In the first square, there are situated two smaller squares (green ones), one with the length of side a, the other with the length of side b. The remaining area is filled with four identical rectangular triangles. In the second square, the triangles are rearranged, the rest of the square is filled with a smaller square with side length c.

Now considering the same area of large squares and the same areas of four rectangular triangles, it can be deduced that the sum of the areas of squares with sides of lengths a and b is equal to the area of square with side length c.

This leads to a well-known formula representing the Pythagorean theorem in the form

$c^2 = a^2 + b^2$.

As has already been said, this is not a true proof, but a graphical interpretation of one situation. If we want to get rid of this deficiency and approach a more general version of the proof, we can use one of the dynamic geometry programs like GeoGebra. In these programs, using sliders and changing parameters, it is possible to switch from a static image to a dynamic figure, where you can experimentally validate the assertion for virtually any of the values a, b, and c (Fig. 2).
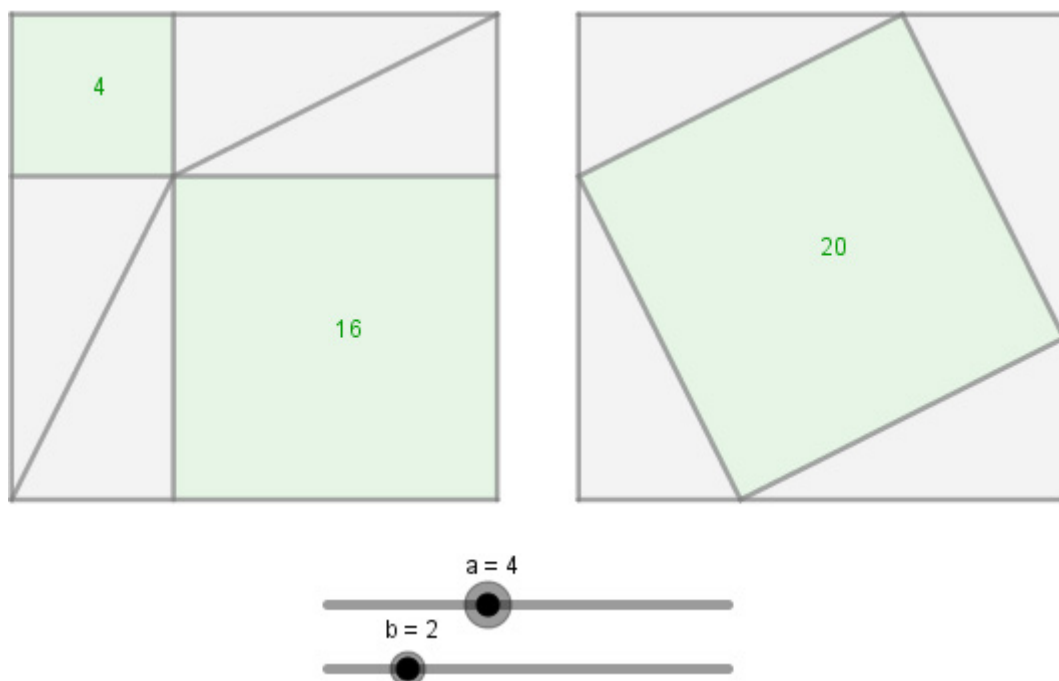

Figure 2. Dynamic figure of proof of Pythagorean theorem

Obviously, using dynamic geometry software offers an element of interaction and dynamics takes geometric proofs (or, more generally, mathematical proofs performed in a geometric way) to another level. At the same time, it can be hoped that software of dynamic geometry can in some cases eliminate certain inaccuracies that the solver can commit if he is to prove only by reasoning and a static sketch. Of course, at this point we have to also point out that when constructing a figure in dynamic geometry programs, it is necessary to consider the behavior of individual objects and the relationships between them in order to avoid possible mistakes resulting from the dynamics of the whole geometric construction.

Here are some examples of erroneous assumptions in geometric proofs. The evidence is supplemented by commentary explaining where the mistake is in the proof, as well as a preview of the problem through dynamic geometry.

441 = 442?

Assertion: Square of side 21 cm has the same contents as the rectangle with sides of 34 cm and 13 cm. (Dubnov, 1954)

Proof: Consider the square of side 21 cm divided into two rectangles of 13 cm × 21 cm and 8 cm × 21 cm. The first rectangle is divided into a pair of identical rectangular trapezoids with base lengths of 13 cm and 8 cm, the second rectangle then into a pair of identical rectangular triangles with lengths of 21 cm and 8 cm.
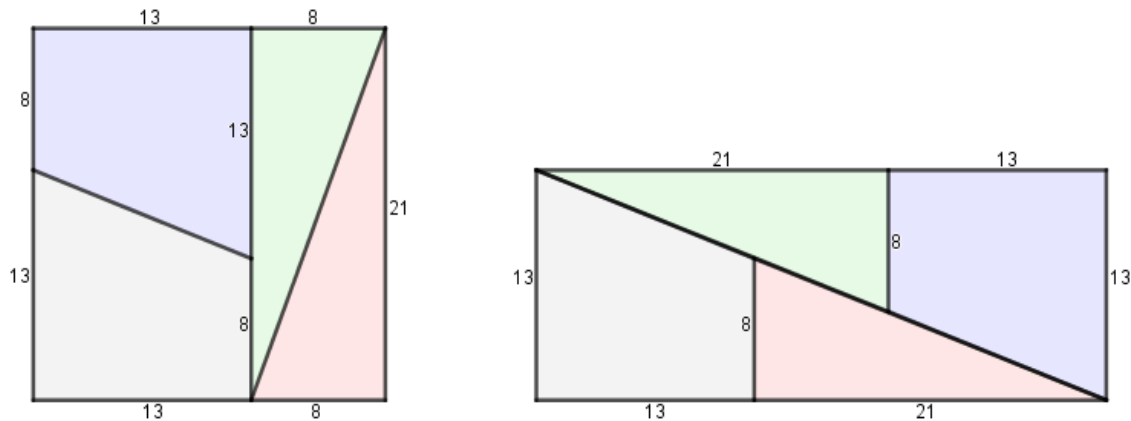

Figure 3. Dividing a square into four parts and their rearranging

As can be seen from Figure 3, a rectangle of sides 13 cm and 34 cm can be obtained by rearranging the parts. All you must do is place a right triangle with legs 8 cm and 21 cm on the respective base of a rectangular trapezoid to form a large rectangular triangle with legs 13 cm and 34 cm. If we place a pair of these triangles with the hypotenuse together, this rectangle is created.

The content of this is 13.34, i.e. 442 cm^2, while the content of the original square was 21^2 = 441 cm^2. All parts that make up a square and a rectangle are identical. The question therefore remains where the excess square centimeter appeared. (Dubnov, 1954)

The answer to this question is quite simple and is not surprising for a reader who has already encountered similar tasks. The whole problem lies in the belief that the two large right triangles are really triangles. We have obtained this idea based on graphical interpretation, but we have no argument for it.

Proof of this assumption could be based, for example, on a comparison of the magnitude of the angles **α** and **β** shown in Figure 4. If the two angles were the same, the geometric figure formed by the composition of the small right triangle and the right trapezoid would be a right triangle.
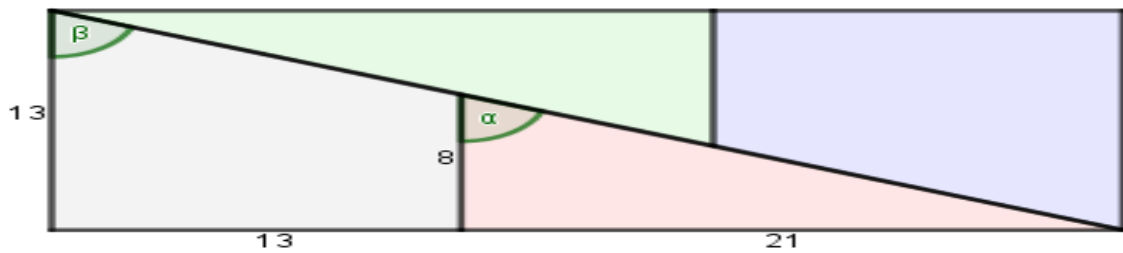
Figure 4. Comparing Angles

Calculation of both angles is easy. For angle α there is equality

$$\tan a = \frac{21}{8}$$

but for angle β there is equality

$$tan\beta = \frac{13}{5}$$,

where we calculate the numerator as the difference of the size of the bases in a rectangular trapezoid.

From comparing fractions

$$\frac{21}{8} > \frac{13}{5}$$

there is also inequality

tan α > tan β.

Here, it is already clear that the angle α is greater than the angle β. Thus, the figure is not a right triangle, but a non-convex quadrilateral. The consequence of this is that the rectangle does not consist of two identical figures (i.e. large right triangles composed of a trapezoid and a smaller right triangle), but of three figures – a pair of identical non-convex quadrilaterals and a parallelogram between them.

If we use the software of dynamic geometry GeoGebra for our examination, it is possible to see the problem from a slightly different point of view – without using calculations as such, but at the same time a bit more clearly.

We construct a rectangle according to the assignment, mark individual geometric figures in it and at the same time use the Polygon tool to create the mentioned parallelogram. If this parallelogram is set to display a value in addition to its name, we will find its content equal to 1 cm², as shown in Figure 5. That is the square centimeter that makes the difference between the content of a square of side 21 cm and the content of a rectangle of sides 13 cm and 34 cm.
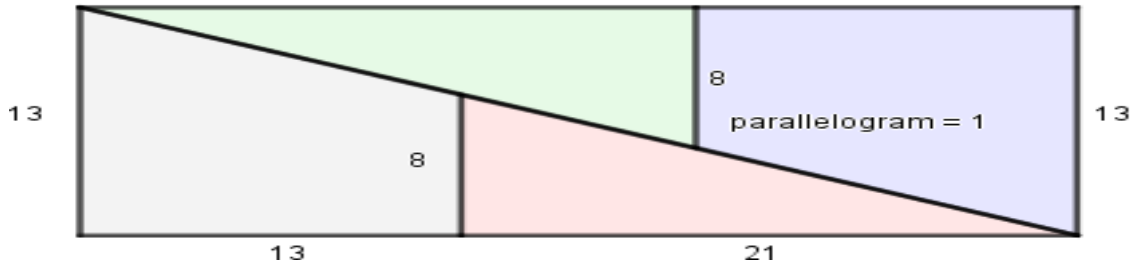
Figure 5. Representation of the Parallelogram and its Contents

Using sliders in a geometric figure makes the situation even clearer. We can experiment with different ways of dividing a given square. It can be divided into two rectangles with dimensions of, for example, 14 cm × 21 cm and 7 cm × 21 cm. Then it is quite clear that after the rearrangement of the created geometric figures, a parallelogram appears in the resulting rectangle. It is now clearly visible (Figure 6) and its content is even 49 cm².
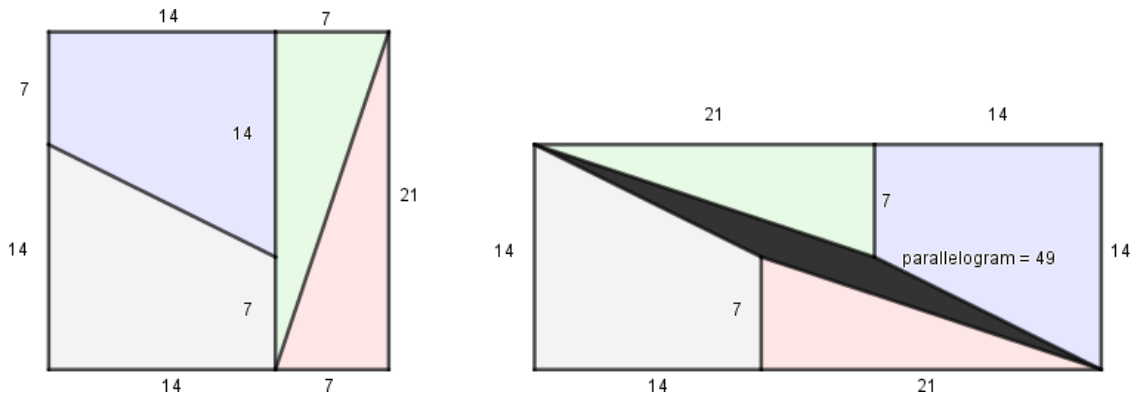

Figure 6. Other division of squares

For the sake of completeness, let's add that this task is actually just one of many variations of one and the same problem called the Curry's paradox, where an object is divided into several parts and these parts are rearranged, with the new object slightly, at first glance, imperceptibly, altered dimensions. Examples of this problem can be found in a task known as "endless chocolate": The chocolate bar is cut into several parts, rearranged, leaving one piece left which can be eaten. With appropriate selection of slices, this rearrangement can then be done several times and one piece can also be eaten each time. An observer who is unaware of this problem can thus feel that chocolate can be eaten without dwindling. The principle is shown in Figure 7. (Gardner, 2018; Thiele, 1985)
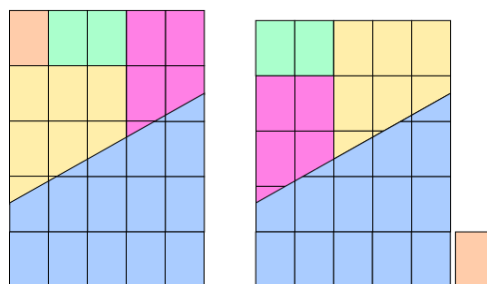

Figure 7. Endless chocolate

Circles with the Same Length

Assertion: All circles have the same length. (Dubnov, 1954)

Proof: To prove the assertion, use one of the practical ways of determining the length of the circle (or wheel circumference). This method is usually used to measure the circumference of a bicycle wheel, which is then set into a cycle computer. The procedure consists of selecting one specific point on the wheel circumference (e.g. bicycle valve) and drawing a mark on the road on which the wheel moves. The markings on the wheel circumference and on the road are initially in the same place and then we roll the wheel. We make a second mark at the point where the point on the wheel circumference again meets the road when rolling. By measuring the distance between the marks on the road we receive wheel circumference.
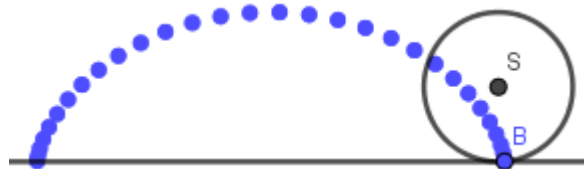

Figure 8. Wheel rolling

This experiment is shown in Figure 8. The path of the selected point B on the wheel circumference is also highlighted. The curve that the point describes during rolling is a cycloid described by parametric equations

$x = r.(t - \sin t)$

$y = r.(1 - \cos t)$,

where r is the radius of the wheel and t is the real parameter.

The distance between two points where point B touches the horizontal line determines the circumference of the wheel (or the length of the circle that is the boundary curve of the wheel).

Now consider a situation where two wheels of different radius are attached to one shaft, as shown in Figure 9. Wheels are fixed on the shaft, i.e. both wheels retain their mutual position when the shaft is rotated.
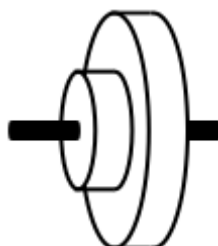

Figure 9. Two Wheels on the Same Shaft

Instead of the road from the previous description, we will now use a pair of parallel rails as a base. Each wheel will roll on one of them. This movement is phased in Figure 10. The figure shows clearly the path of point *A* on the larger wheel and the circumference of that wheel. However, it is worth noting the individual positions of point *A'*. As can be seen, the distance between two points where point *A'* touches the rail is the same as the distance between two points of point *A*. This implies that both wheels must have the same circumference.
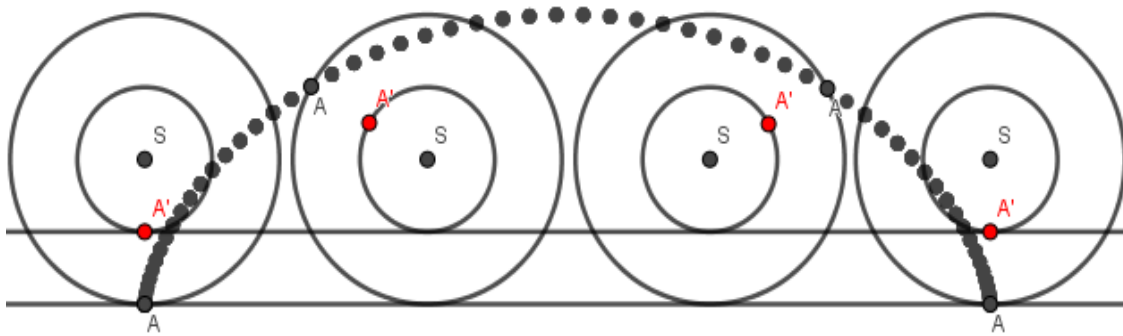

Figure 10. Rolling wheels on rails

Finally, we can generalize the previous statement and convert it back to the geometric level. Thus, it can be argued that the length of the circle is given regardless of the radius of the circle. In other words, two arbitrarily selected circles of different radii are always the same length. (Dubnov, 1954)

This statement is, of course, erroneous, as every reader knows from experience. It is enough to find a problematic passage in the chain of considerations to refute the statement. First, it should be noted that the whole problem is a kinematic issue rather than a geometric one. However, it is still possible to perform a geometric construction of a dynamic figure in software GeoGebra to detect the mistake.

For this purpose, we construct a pair of parallel lines showing both the rails, two concentric circles (in our case the larger circles have a double radius), each with one point *A* and *A'*, and use the slider at the same time. By using it we control the movement of circles in the direction of the lines and at the same time we use it to move the marked points. The movement consists of a displacement in the direction of the lines and at the same time a rotation around the center of both circles. The situation constructed in this way corresponds to the considerations described above, from which we have deduced the same length of both circles.

Now let's use the function to display the track of a moving object. We will focus on the traces of movement of both points *A* and *A'*. It is clear from Figure 11 that while point *A* leaves a trail corresponding to the cycloid while moving, point *A'* draws a different track.
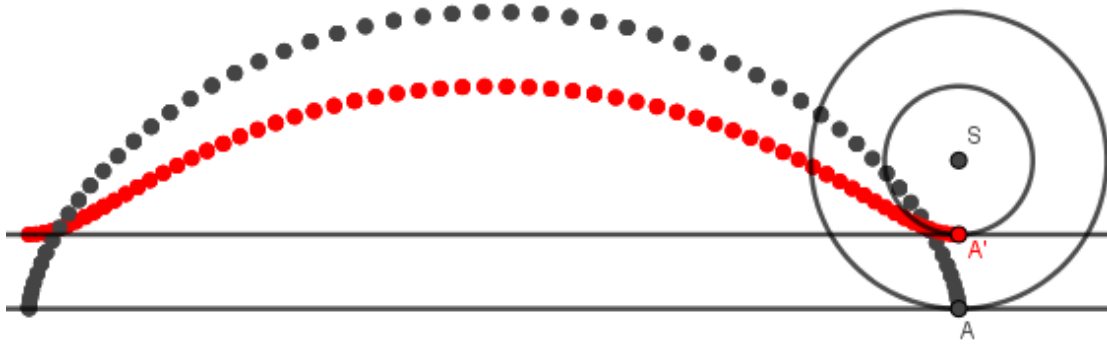
Figure 11. Traces of Marked Points *A*, *A'*

It is a so-called shortened cycloid whose parametric expression is given by a pair of equations

$$x = r.t - a.\sin t$$

$$y = r.1 - a.\cos t,$$

where *r* =|*SA*| (i.e. the radius of the larger circle) a *a* =|*SA'*| (i.e. the radius of the smaller circle), so the inequality *a* < *r*.

Since point *A'* did not describe the expected cycloid in its movement, but rather a shortened cycloid, we can assume that this is where the problem is being sought. The movement of point *A'* as it is modeled in the dynamic figure does not correspond to what it should actually look like.

Suppose we found a mistake, and at the same time we know the form of the output (the cycloid trace) we are looking for. With a little bit more work, we are able to complete the figure with another point *B*. When the track is switched on, it renders the cycloid again, but with half the period compared to the cycloid given by point *A*. The result is therefore a successful rebuttal of the false proof, since the length of one circle with half the radius relative to the radius of the other circle is also half.
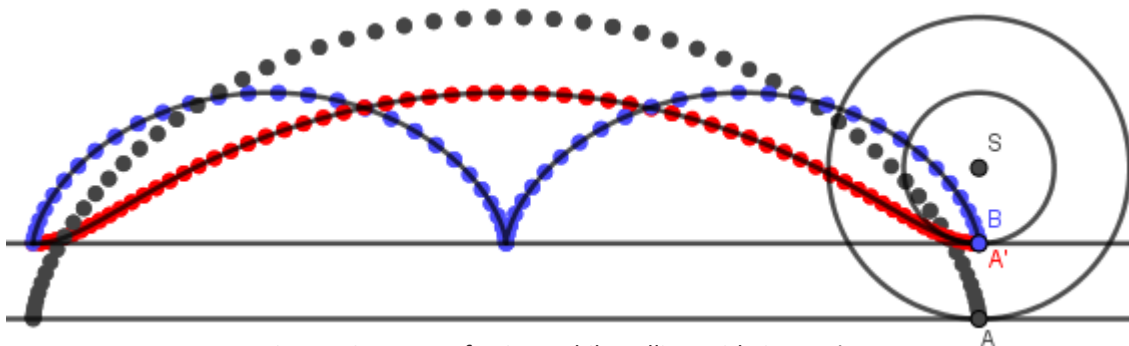

Figure 12. Traces of Points While Rolling With Correction

Finally, let's move from geometry back to kinematics and specify the mistake from the physical point of view. This mistake must be found in the fixed position of the two wheels on the same shaft, where the two wheels maintain their relative position when rotating. When the wheels roll on the rails, one of the wheels slips. Specifically, when rolling

a larger wheel without slipping, this is a negative slip of the smaller wheel, thereby increasing the distance between the two marks on the appropriate rail. Otherwise, rolling the smaller wheel without slipping would be a positive slip of the larger wheel, which in turn would reduce the distance between the marks.

$\pi = 2$

Assertion: The number π is equal to 2. (Dubnov, 1954)

Proof: Let's have a line *AB* over which we construct a semicircle. The length of this semicircle is obtained from the formula for calculating the circumference of circle *C* = $2\pi.r = \pi.d$ and it is equal to $\pi/2|AB|$. Then we halve the line *AB* and construct a semicircle above each of its halves, with the left semicircle placed above the line and the right semicircle below the line. This creates a "wavy" line resembling a portion of the sine graph, and for each semicircle, its length is half that of the original large semicircle above the line *AB*. The total length of the "wavy" line is therefore again $\pi/2|AB|$.

If we continue to halve existing lines and construct new semicircles in the same way, a "wavy" line is created, which is increasingly closer to line *AB*. This represents the limit of the "wavy" line for the number of halves going to infinity. However, the length of the "wavy" line is still equal to $\pi/2|AB|$. The size of the limit and line *AB* must also be the same. From equality $\pi/2|AB| = |AB|$ we obtain equality $\pi = 2$. (Dubnov, 1954)
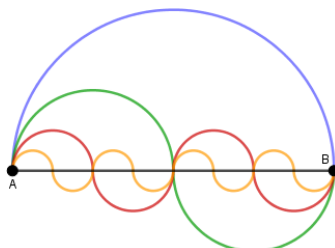

Figure 13. Softening the "Wavy" Line

The problem with this inference with a totally nonsensical consequence lies in a misunderstanding of the term "limit". Or more precisely, in the possible ambiguity of the term "limit". These two different senses may but also may not be equivalent and thus interchangeable.

On the one hand, it is a "wavy" line, which is essentially a sequence consisting of semicircles. The limit of this sequence is indeed the line segment *AB*. As the number of arcs increases, the arcs of the semicircles approach unlimitedly. On the other hand, it is a numerical sequence, the members of which are the lengths of the individual semicircles and whose sum is to be unlimited close to a certain fixed limit value.

The reader must not be fooled by the fact that with a very fine line segment, and

therefore with very small semicircles, the "wavy" line practically does not differ from the line segment *AB* itself. To solve this problem in GeoGebra simply zoom enough.

At the same time, consider the following: The difference between the lengths of the semicircle and the line segment above which the semicircle is circumscribed is always non-zero (no matter how short the line is). With very small dimensions of the semicircles, these differences are slight, but they occur in very large quantities. However, even the smallest values can give an arbitrarily large sum in enough quantity.

In other words, the sequence of semicircles that consist of a "wavy" line is close to line *AB*, but the sum of their lengths has nothing in common with the line length as its limit.

As in the previous cases, let's try to use geometric software to illustrate the situation. In software GeoGebra, we construct a line *AB* and a default semicircle above it. Next, using slider and the Sequence command we create a lists of points and small semicircles. Moving the slider increases the number of semicircles to soften the "wavy" line at the same time.

As shown in Figure 14, irrespective of the change in the number of semicircles, the length of the "wavy" line is still equal to π (except for some small rounding variations in program calculations, which is set to 2 decimal places by default).
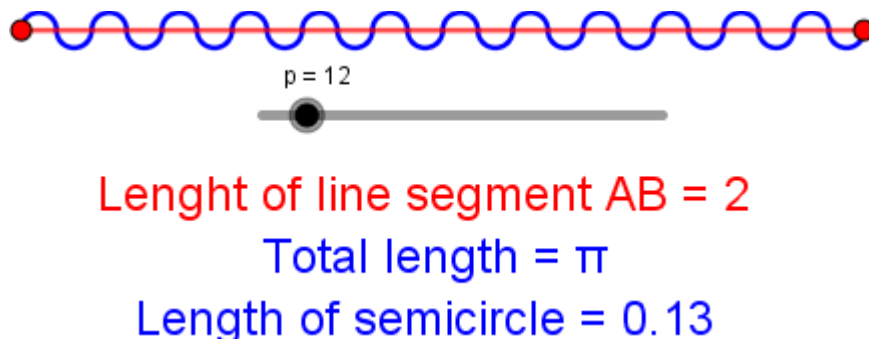

Figure 14. The Length of the "Wavy" Line

Conclusion

In this paper, we have presented some geometric evidence, which at first glance may seem correct, but lead to totally erroneous conclusions. Apart from our logical considerations, we relied mainly on the software of dynamic geometry GeoGebra, which proved to be a proficient helper in refuting the erroneous conclusions. Even though it is not primarily intended for proving geometric theorems. Nevertheless, as shown in the previous examples, it can also be used successfully in this area by allowing the user to perform demonstrations and experiments through interactivity and the ability to dynamically manipulate geometric objects. These can then clearly help to avoid hasty and erroneous conclusions that can be reached, for example, by elementary and high school pupils and students.

References

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